

Tree-Level Unitarity and Renormalizability in Lifshitz Scalar Theory

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Abstract

We study unitarity and renormalizability in the Lifshitz scalar field theory, which is characterized by an anisotropic scaling between the space and time directions. Without the Lorentz symmetry, both the unitarity and the renormalizability conditions are modified from those in relativistic theories. We show that for renormalizability, an extended version of the power counting condition is required in addition to the conventional one. The unitarity bound for S-matrix elements also gives stronger constraints on interaction terms because of the reference frame dependence of scattering amplitudes. We prove that both unitarity and renormalizability require identical conditions as in the case of conventional relativistic theories.

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1 Introduction

A field theory has to possess both renormalizability and unitarity to be a well-defined quantum theory. These properties, which may be intimately related to each other, are believed to provide identical constraints on consistent interactions. This idea motivated the study of the equivalence between the renormalizability and the tree-level unitarity in gauge theories [1, 2, 3]. It has been shown that a similar equivalence also holds for gravity theories [4] and no counter-example is known in relativistic field theories¹. It is worth checking whether the equivalence holds true for more generic field theories, such as non-relativistic theories. The purpose of this paper is to investigate the equivalence between renormalizability and unitarity in Lifshitz-type theories [5], which are characterized by the Lifshitz scaling

$$t \rightarrow b^z t, \quad x^i \rightarrow b x^i \quad (i = 1, \dots, d). \quad (1)$$

Due to this anisotropic scaling property, Lifshitz-type theories have improved ultraviolet (UV) behaviors but lack the Lorentz symmetry.

Lifshitz-type theories have gotten a lot of attention in cosmology [7] and some their quantum properties are also investigated [8, 9, 10, 11]. Moreover, by introducing the idea of the Lifshitz scaling to gravity theory, Hořava has constructed a power-counting renormalizable gravity theory, which is called the Hořava-Lifshitz (HL) gravity [12]. The application of the HL gravity is widely studied in cosmology (see [13] for a review), such as the emergence of the dark matter as an integration constant [14] and so on. In contrast, the quantum properties, such as unitarity and renormalizability, of HL gravity still remain uncertain.

In this paper, we study the quantum aspects of the Lifshitz scalar field theory, which can be viewed as a toy model of the HL gravity. The lack of the Lorentz symmetry gives rise to significant modifications to both renormalizability and unitarity. We show that due to the modification of the scaling dimensions of fields, an extended version of the power counting argument is necessary for renormalizability. As for tree-level unitarity, the reference frame dependence of scattering amplitudes gives rise to new modified constraints, which can be stronger than those derived only from the high energy center-of-mass (COM) scattering amplitudes. We derive the extended power-counting-renormalizability (PRC) conditions and the tree-level unitarity constraints on self-interactions of a single Lifshitz scalar field. Our result shows that the equivalence holds true also for the extended versions of renormalizability and unitarity.

This paper is organized as follows. In Sec. 2, we derive the extended PCR constraints, which

¹ Theories with negative norm states (ghost states), such as in higher curvature theory, are excluded in the discussion of the equivalence between unitarity and renormalizability.

are necessary conditions for renormalizability. After reviewing the unitarity bound in Sec. 3, we derive unitarity conditions for high-energy two-body scattering amplitudes of the Lifshitz scalar in Sec. 4. In Sec. 5, unitarity constraints on quartic and cubic interactions are derived from the unitarity conditions for the high-energy scattering amplitude. We see that the unitarity constraints are equivalent to the extended PCR constraints derived in Sec. 2. In Sec. 6, s-channel scattering amplitudes with on-shell intermediate particles are discussed. By correctly taking into account the resonance effects, we show that interactions satisfying the extended PCR conditions do not violate the unitarity bound. Finally, we give the summary and discussion in Sec. 7.

2 Renormalizability Condition

In this section, we derive necessary conditions for the renormalizability of the Lifshitz scalar field theory. Since scalar fields can have negative scaling dimensions, we have to carefully use the power counting argument to check the renormalizability of the theory. We show that renormalizable interactions have to satisfy an extended version of the PCR condition.

The generic form of the action for a Lifshitz scalar field ϕ is given by

$$S = \int dt d^d x \left[\frac{1}{2} \phi \left\{ \partial_t^2 - f(-\Delta) \right\} \phi + \mathcal{L}_{int} \right], \quad \Delta := \partial^i \partial_i, \quad (2)$$

where $f(-\Delta) = (-\Delta)^z + \dots$ is a polynomial of degree z and the interaction Lagrangian \mathcal{L}_{int} will be specified shortly. The dispersion relation takes the form

$$E = \sqrt{f(p^2)}, \quad (3)$$

where E and p are the energy and the magnitude of the momentum, respectively. This dispersion relation becomes $E \approx p^z$ in the UV regime and is consistent with the Lifshitz scaling (1). Since the asymptotic behavior of propagators is given by

$$\frac{1}{E^2 - f(p^2)} \approx \frac{1}{E^2 - p^{2z}}, \quad (4)$$

UV divergences of loop diagrams are milder than those in relativistic theories. The scaling dimension of ϕ can be read from the quadratic term in (2):

$$[\phi] = \frac{d - z}{2}, \quad (5)$$

where we have used the convention such that $[E] = z$ and $[p] = 1$. In the following, we will consider the scaling law (1) for arbitrary values of z including $z > d$, for which $[\phi]$ is negative. We will see that the power counting argument should be modified for $[\phi] < 0$ ($z > d$). In our previous paper [8], we have investigated that the Lifshitz scalar theory with $z = d$, for which $[\phi] = 0$, already presents unconventional features concerning its renormalizability.

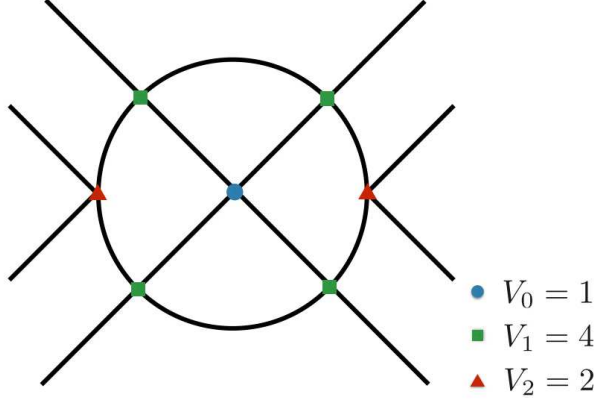


Fig. 1: A loop diagram with $n = 4$, $V_0 = 1$, $V_1 = 4$, $V_2 = 2$ ($N = 8$).

2.1 Extended Power-Counting-Renormalizability Condition

Let us consider the following generic n -point interaction term

$$S_{int} = \lambda \int dt d^d x (\partial_x^{a_1} \phi) (\partial_x^{a_2} \phi) \cdots (\partial_x^{a_n} \phi), \quad (6)$$

where a_1, \dots, a_n are non-negative integers. Each ∂_x denotes any of the spatial derivatives ∂_i ($i = 1, \dots, d$) and we suppose that this interaction term is designed to keep the d -dimensional spatially rotational symmetry $O(d)$. Without loss of generality, we can rearrange $\partial_x^{a_i} \phi$ ($i = 1, \dots, n$) so that $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$. The dimension of the coupling constant λ is given by

$$[\lambda] = z + d - \sum_{l=1}^n (a_l + [\phi]). \quad (7)$$

In relativistic theories, an interaction is renormalizable if it has a coupling constant with non-negative dimension: $[\lambda] \geq 0$. On the other hand, in the Lifshitz scalar theory, $[\lambda]$ has to satisfy a more strict condition, which can be derived as follows.

For a one-particle irreducible (1PI) loop diagram, let V_e ($e = 0, 1, \dots, n-2$) be the numbers of vertices with e external lines. Note that $V_{n-1} = V_n = 0$ for any 1PI loop diagrams. An example with $n = 4$ is shown in Fig. 1. The contribution of the vertex with e external lines takes the form

$$\text{Vertex for Eq. (6)} = \lambda (q_1^{a_1} \cdots q_e^{a_e}) (p_1^{a_{e+1}} \cdots p_{n-e}^{a_n}) + (\text{permutations w.r.t. } a_i), \quad (8)$$

where p_i and q_i are internal and external momenta, respectively. Since we have set $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$, the first term is the dominant contribution in the limit $p_i \rightarrow \infty$. From this expression, the leading order part of the loop integral can be estimated as

$$\text{1PI loop diagram} \approx \int (dE d^d p)^L \left(\frac{1}{E^2 - p^{2z}} \right)^P \prod_{e=0}^{n-2} \left(\prod_{l=e+1}^n p^{a_l} \right)^{V_e}, \quad (9)$$

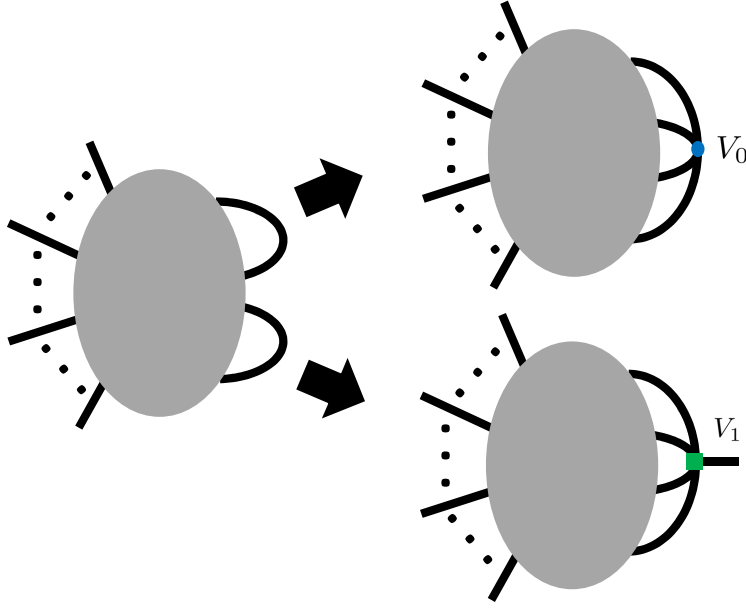


Fig. 2: Adding a vertex to divergent diagram: The left diagram is a divergent loop diagram. The upper and lower arrows show the operations of adding V_0 and V_1 vertices, respectively. The upper operation does not change the structures of external line, while the lower operation creates an additional external line.

where L and P are the numbers of loops and internal propagators, respectively. This implies that the degree of divergence of the loop diagram is

$$D = L(z + d) - 2zP + \sum_{e=0}^{n-2} V_e \sum_{l=e+1}^n a_l. \quad (10)$$

Since each vertex has $n - e$ internal lines and each internal line is connected to two vertices, the number of the internal propagators P is given by

$$P = \frac{1}{2} \sum_{e=0}^{n-2} V_e (n - e). \quad (11)$$

The number of loop integrals L can be identified with the number of internal momenta which are not determined by the momentum conservation law. Since there is one delta function at each vertex and one of them is for the overall momentum, the number of loop L is given by

$$L = P - \sum_{e=0}^{n-2} V_e + 1. \quad (12)$$

Eliminating L and P from Eq. (10) and using Eqs. (5) and (7), we obtain

$$D = z + d - \sum_{e=0}^{n-2} V_e ([\lambda] + d_e), \quad d_e := \sum_{l=1}^e (a_l + [\phi]). \quad (13)$$

For a renormalizable interaction, N -point diagrams have to be finite for sufficiently large N so that all UV divergences can be eliminated by a finite number of counter terms. In other

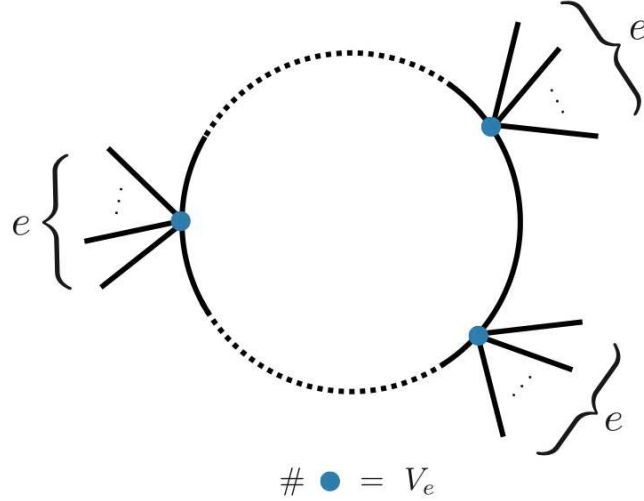


Fig. 3: A loop diagram constructed only with vertices satisfying $[\lambda] + d_e = 0$: Regardless of the number of vertices, this loop diagram diverges with degree $D = z + d$.

words, D has to be negative for loop diagrams with sufficiently large numbers of external lines $N = \sum_e e V_e$. If the contribution of each vertex $[\lambda] + d_e$ is negative for some e , the degree D does not become negative as $V_e \rightarrow \infty$. It means that there are infinitely many UV divergent graphs with different N . Such divergences cannot be absorbed by a finite number of counter terms, and hence this type of interactions is non-renormalizable.

Next, we consider interactions with $[\lambda] + d_e = 0$ for some $e \geq 1$. The corresponding vertices in a loop diagram do not change the value of D but increase the number of external lines (see Fig. 2). Using this type of interactions, we can construct an infinite number of divergent loop diagrams with different numbers of external legs N as shown in the example in Fig. 3. Hence this type of interactions is also non-renormalizable. For $e = 0$, in contrast, interactions with $[\lambda] + d_0 = 0 (= [\lambda])$ are not excluded, since the addition of the corresponding vertices does not change the number of the external lines (see Fig. 2). Therefore, as in the case of the conventional scalar field theory, the number of the required counter terms does not increase and hence interaction terms with $[\lambda] = 0$ can be renormalizable.

In summary, the power-counting argument gives the following conditions for the coupling constant λ of a renormalizable interaction:

$$[\lambda] \geq 0, \quad (\text{for } e = 0), \quad (14)$$

$$[\lambda] > -d_e, \quad (\text{for all } e = 1, 2, \dots, n-2). \quad (15)$$

The first condition is the conventional PCR condition. In addition, the other conditions (15) are also required for the renormalization in the Lifshitz scalar field theory. We call these conditions

the extended PCR. For $z \leq d$, the dimension of the scalar field $[\phi] = (d - z)/2$ is non-negative and hence d_e ($e = 1, \dots, n - 2$) are also non-negative. In such a case, the conditions (15) are automatically satisfied as long as the conventional PCR condition $[\lambda] \geq 0$ holds. On the other hand, d_e ($e = 1, \dots, n - 2$) can be negative for $z > d$, so that $[\lambda]$ should satisfy the most strict of the conditions in (15).

It is convenient to rewrite Eqs. (14) and (15) into the following equivalent conditions:

$$\sum_{l=1}^n (a_l + [\phi]) \leq z + d, \quad (16)$$

$$\sum_{l=e+1}^n (a_l + [\phi]) < z + d, \quad (\text{for all } e = 1, \dots, n - 2). \quad (17)$$

The latter condition implies that the dimension of $\partial_x^{a_{e+1}} \phi \cdots \partial_x^{a_n} \phi$ ($e = 1, \dots, n - 2$) should be less than $z + d$.

To show the consistency of the extended PCR condition, it is important to check the renormalizability of counter terms introduced to cancel the UV divergences. In Appendix A.1, we show that the counter terms for renormalizable interactions also satisfy the extended PCR condition. If the dimension of the field is negative, there are an infinite number of terms satisfying the extended PCR. However, we show in Appendix A.2 that if the number of terms initially introduced in the bare action is finite, the theory can be renormalized with a finite number of counter terms.

2.2 Example

We illustrate the importance of the extended PCR condition (15) by taking an example. Due to the Lifshitz scaling, the scalar field has a negative dimension for $z > d$. In such cases, there can be interactions that have a coupling constant λ with non-negative dimension but do not satisfy the condition (15). By using the corresponding vertices, we can construct infinitely many divergent diagrams even if the conventional PCR condition (14) is satisfied.

We consider an example with $d = 3$ and $z = 5$, in which the dimension of the scalar field is $[\phi] = -1$. The quartic interaction term (corresponding to $a_1 = a_2 = 0$, $a_3 = a_4 = 6$)

$$S_4 = \lambda \int dt d^3x \phi^2 (\Delta^3 \phi)^2, \quad (18)$$

has the dimensionless coupling constant λ , and thus this interaction satisfies the conventional PCR condition $[\lambda] \geq 0$. However, since $a_3 + a_4 + 2[\phi] = 10 > 8 = z + d$, the extended PCR (17) with $e = 2$ is not satisfied.

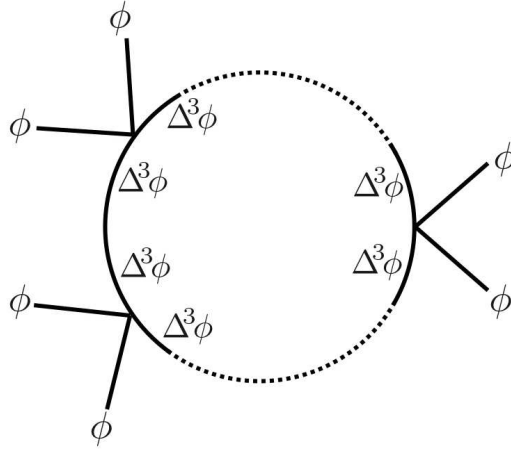


Fig. 4: $2n$ -point one-loop diagram with n 4-point vertices: The dotted parts contain the same vertices as those drawn in the figure. ϕ and $\Delta^3\phi$ indicate that the neighboring lines correspond to ϕ and $\Delta^3\phi$ in (18).

Let us focus on the one-loop diagram with $2n$ external legs shown in Fig.4. Its leading behavior is given by the term for which two $(\Delta^3\phi)$'s at each vertex are contracted with those in the two neighboring vertices. Thus, each vertex gives the leading behavior $\approx p^{12}$ for large loop momentum $p \rightarrow \infty$. Since this diagram consists of n four-point vertices and n internal propagators, the degree of divergence is estimated as

$$\int d\omega d^3p \left(\frac{1}{\omega^2 - p^{10}} \right)^n (p^{12})^n \sim \Lambda^{8+2n}, \quad (19)$$

where Λ is a UV cutoff with $[\Lambda] = 1$. For any n , this loop integral diverges as $\Lambda \rightarrow \infty$, and thus the interaction (18) is not renormalizable.

The reason why the quartic action (18) is non-renormalizable can be explained as follows. In the leading contribution of each vertex, the part of the operator (18) involved in the loop integral is $(\Delta^3\phi)^2$, whose dimension is $[(\Delta^3\phi)^2] = 10$. Since the dimension of ϕ is negative, this is higher than the dimension of the whole operator: $[\phi^2(\Delta^3\phi)^2] = 8$. In general, operators with dimension $8(= z + d)$ represent marginal interactions while those with higher dimensions are non-renormalizable. Since the operator $(\Delta^3\phi)^2$, which is relevant to the loop integral, has higher dimensions, the interaction (18) is non-renormalizable. This peculiar phenomenon occurs because, if the dimension of the field $[\phi]$ is negative, a part of interaction term $(\Delta^3\phi)^2$ can have higher dimensions than whole interaction term $\phi^2(\Delta^3\phi)^2$; the same phenomenon does not occur in the conventional relativistic theory, in which ϕ always has a non-negative dimension.

3 Unitarity Bound

In this section, we briefly review the unitarity bound. It is a necessary condition for scattering amplitudes derived from the unitarity of the S-matrix

$$SS^\dagger = \mathbf{1}. \quad (20)$$

Decomposing the S-matrix as $S = \mathbf{1} + iT$, we can rewrite the unitarity condition into

$$-i(T - T^\dagger) = TT^\dagger. \quad (21)$$

We define the scattering amplitude $\mathcal{M}(i \rightarrow f)$ from the matrix element of T by factoring out the delta functions which indicate the energy-momentum conservation:

$$\langle f|T|i\rangle = \delta(E_i - E_f)\delta^d(\mathbf{p}_i - \mathbf{p}_f) \mathcal{M}(i \rightarrow f). \quad (22)$$

Let $\{|X\rangle\}$ be a complete orthonormal basis of the Hilbert space:

$$\sum_X |X\rangle\langle X| = 1. \quad (23)$$

Then, Eq.(21) can be rewritten as

$$-i[\mathcal{M}(i \rightarrow f) - \mathcal{M}(f \rightarrow i)^*] = \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X) \mathcal{M}(i \rightarrow X) \mathcal{M}(f \rightarrow X)^*, \quad (24)$$

where $E(= E_i = E_f)$ and $\mathbf{p}(= \mathbf{p}_i = \mathbf{p}_f)$ are the total energy and momentum respectively. In particular, for identical initial and final states, the relation reduces to the optical theorem

$$2\text{Im}\mathcal{M}(i \rightarrow i) = \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X) |\mathcal{M}(i \rightarrow X)|^2, \quad (25)$$

where \sum_X denotes the sum with respect to all possible intermediate states. In the following, we ignore unimportant numerical factors.

The n -particle Hilbert space is spanned by the standard asymptotic momentum eigenstates $|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle$, which we normalize as

$$\int \prod_{j=1}^n \frac{d^d p_j}{2E_j} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \langle \mathbf{p}_1, \dots, \mathbf{p}_n| = 1, \quad (26)$$

with $E_j = \sqrt{f(p_j^2)}$. In the discussion of the unitarity bound, it is convenient to use another basis $|E, \mathbf{p}, l\rangle$ with a discrete label l defined as follows. First, let us consider the subspace of the n -particle momentum space specified by fixed values of the total energy and momentum

$$\sum_{j=1}^n E_j = E, \quad \sum_{j=1}^n \mathbf{p}_j = \mathbf{P}. \quad (27)$$

This constant-energy-momentum space, which we denote by $\mathcal{C}_{(E,\mathbf{P})}$, is a compact subspace of $\mathbb{R}^{n(d+1)}$. Introducing a set of orthonormal functions $\{h_l(\mathbf{p}_j)\}$ on $\mathcal{C}_{(E,\mathbf{P})}$, we can define $|E, \mathbf{P}, l\rangle$ by

$$|E, \mathbf{P}, l\rangle = \int d\Pi_n h_l(\mathbf{p}_j) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle, \quad (28)$$

where $d\Pi_n$ is the standard phase space volume element on $\mathcal{C}_{(E,\mathbf{P})}$:

$$d\Pi_n \equiv \prod_{j=1}^n \frac{d^d p_j}{2E_j} \delta(E_1 + \dots + E_n - E) \delta^d(\mathbf{p}_1 + \dots + \mathbf{p}_n - \mathbf{P}). \quad (29)$$

Let $\mathcal{M}(E, \mathbf{P}; l \rightarrow l')$ be the scattering amplitude defined by

$$\langle E, \mathbf{P}, l | T | E', \mathbf{P}', l' \rangle = \delta(E - E') \delta^d(\mathbf{P} - \mathbf{P}') \mathcal{M}(E, \mathbf{P}; l \rightarrow l'). \quad (30)$$

For this scattering amplitude, the unitarity bound can be obtained from Eq.(25):

$$\begin{aligned} |\mathcal{M}(E, \mathbf{P}; l \rightarrow l)| &\geq |\text{Im} \mathcal{M}(E, \mathbf{P}; l \rightarrow l)| \\ &= \sum_{l'} |\mathcal{M}(E, \mathbf{P}; l \rightarrow l')|^2 \geq |\mathcal{M}(E, \mathbf{P}; l \rightarrow l')|^2. \end{aligned} \quad (31)$$

Here, in the last expression, we have picked up a specific state l' from the sum over any possible intermediate states. If we choose $l' = l$, the unitary bound (31) implies that $|\mathcal{M}(E, \mathbf{P}; l \rightarrow l)| < \text{const.}$ From this inequality and Eq. (31), it follows that

$$|\mathcal{M}(E, \mathbf{P}; l \rightarrow l')| \leq \text{const}, \quad \forall l, l'. \quad (32)$$

Therefore, unitarity is violated if the scattering amplitudes between the normalized discrete states (28) are not bounded from above.

For later use, we rewrite $\mathcal{M}(E, \mathbf{P}; l \rightarrow l')$ in terms of the scattering amplitudes between the standard momentum eigenstates:

$$\begin{aligned} &\langle \mathbf{p}_1, \dots, \mathbf{p}_{n_i} | T | \mathbf{k}_1, \dots, \mathbf{k}_{n_f} \rangle \\ &= \delta \left(\sum_{j=1}^{n_i} E_j - \sum_{m=1}^{n_f} E_m \right) \delta^d \left(\sum_{j=1}^{n_i} \mathbf{p}_j - \sum_{m=1}^{n_f} \mathbf{k}_m \right) M(\mathbf{p}_1, \dots, \mathbf{p}_{n_i} \rightarrow \mathbf{k}_1, \dots, \mathbf{k}_{n_f}). \end{aligned} \quad (33)$$

By using the relation (28), the amplitude $|\mathcal{M}(E, \mathbf{P}; l \rightarrow l')|$ can be rewritten as

$$\mathcal{M}(E, \mathbf{P}; l \rightarrow l') = \int d\Pi_{n_i}(\mathbf{p}_j) d\Pi_{n_f}(\mathbf{k}_m) h_l(\mathbf{p}_j) \bar{h}_{l'}(\mathbf{k}_m) M(\mathbf{p}_1, \dots, \mathbf{p}_{n_i} \rightarrow \mathbf{k}_1, \dots, \mathbf{k}_{n_f}). \quad (34)$$

The unitarity bound should be satisfied for arbitrary choices of the orthonormal basis $\{h_l(\mathbf{p}_j)\}$. This implies that scattering amplitudes have to be bounded from above for any normalized initial and final states, i.e. for any normalized functions $h_l(\mathbf{p}_j)$ and $h_{l'}(\mathbf{k}_j)$ defined on $\mathcal{C}_{(E,\mathbf{P})}$.

4 Unitarity Bound in Lifshitz Scalar Theory

In this section, we derive high-energy behaviors of scattering amplitudes in the Lifshitz scalar theory. In the following, we restrict ourselves to two-particle scattering. In relativistic theories, it is sufficient to consider scattering amplitudes in the center-of-mass (COM) frame since we can use the Lorentz boost to change the reference frame. In contrast, without the Lorentz symmetry, we also need to consider situations where the total momentum is non-zero. Such scattering processes are not related to those in the COM frame and may give different unitarity constraints on interaction terms.

Since we are interested in high-energy scattering amplitudes with a large total momentum \mathbf{P} , we set

$$E \approx P^z, \quad (35)$$

and take the limit $P := |\mathbf{P}| \rightarrow \infty$. In this setting, there are two typical states which have different high energy behaviors:

- a) Two-particle state whose momenta have the same magnitude $|\mathbf{p}_1| = |\mathbf{p}_2|$.
- b) Two-particle state with momenta $\mathbf{p}_1 = 0$ and $\mathbf{p}_2 = \mathbf{P}$.

For **a)**, the energies of both particles approach infinity as in the case of the COM scattering. For **b)**, only one particle has high energy and the other is at rest (laboratory system), so that the situation is very different from the COM case.

We construct two normalized states $|E, \mathbf{P}, l\rangle$ ($l = \alpha, \beta$) which possess the properties of the two states **a)** and **b)** respectively. The first one, denoted by $|\alpha\rangle$, is the state for which the energies of two particles approach infinity as $P \rightarrow \infty$. It can be obtained by defining the function $h_{l=\alpha}(\mathbf{p}_1, \mathbf{p}_2)$ as

$$h_{\alpha}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{\sqrt{N_{\alpha}(P)}} \times \begin{cases} 1 & (||\mathbf{p}_1| - |\mathbf{p}_2|| \leq P/2) \\ 0 & (||\mathbf{p}_1| - |\mathbf{p}_2|| > P/2) \end{cases}. \quad (36)$$

The normalization factor N_{α} is given by

$$N_{\alpha}(P) = \int_{I_{\alpha}} \frac{d^d p_1}{2E_1} \frac{d^d p_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}), \quad (37)$$

where the domain of integration is

$$I_{\alpha} = \left\{ (\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid ||\mathbf{p}_1| - |\mathbf{p}_2|| \leq P/2 \right\}. \quad (38)$$

By a simple dimensional analysis², we can estimate the asymptotic behavior of N_α in the limit $P \rightarrow \infty$ as (see Appendix B for more details on the phase space integrals)

$$N_\alpha(P) \approx P^{d-3z}. \quad (39)$$

Based on the definition of the discrete normalized state Eq. (28), we define the state $|\alpha\rangle$ by

$$|\alpha\rangle = \frac{1}{\sqrt{N_\alpha(P)}} \int_{I_\alpha} \frac{d^d \mathbf{p}_1}{2E_1} \frac{d^d \mathbf{p}_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) |\mathbf{p}_1, \mathbf{p}_2\rangle. \quad (40)$$

The second one, denoted by $|\beta\rangle$, is the state for which only one particle has high energy in the limit $P \rightarrow \infty$. The normalized function

$$h_\beta(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{\sqrt{N_\beta(P)}} \times \begin{cases} 1 & (\text{for } |\mathbf{p}_1| \leq \epsilon) \\ 0 & (\text{for } |\mathbf{p}_1| > \epsilon) \end{cases}, \quad (41)$$

gives such a state, where ϵ is a constant satisfying $\epsilon \ll P$. The normalization factor $N_\beta(P)$ is defined as in Eq. (37) with the domain of integration

$$I_\beta = \{(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid |\mathbf{p}_1| \leq \epsilon\}. \quad (42)$$

The state $|\beta\rangle$ is defined by

$$|\beta\rangle = \frac{1}{\sqrt{N_\beta(P)}} \int_{I_\beta} \frac{d^d \mathbf{p}_1}{2E_1} \frac{d^d \mathbf{p}_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) |\mathbf{p}_1, \mathbf{p}_2\rangle. \quad (43)$$

Note that in the high-energy limit, this state exists only when $E = P^z + \mathcal{O}(P^{z-1})$. We can estimate the asymptotic behavior of the normalization factor $N_\beta(P)$ as follows (see Appendix B for more details). The second particle has a large momentum $|\mathbf{p}_2| \approx P$, so that $d^d p_2$ and E_2 behave as P^d and P^z , respectively. On the other hand, $d^d p_1$ and E_1 do not depend on P since \mathbf{p}_1 is kept small in the high energy limit. After eliminating the delta function $\delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \approx P^{-d}$, the terms of order P^z in the argument of $\delta(E_1 + E_2 - E)$ cancel with each other and those of order P^{z-1} become dominant. Therefore, $\delta(E_1 + E_2 - E)$ is of order $P^{-(z-1)}$, and hence N_β behaves as

$$N_\beta(P) \approx P^{-2z+1}, \quad (44)$$

in the high energy limit.

Given the two states $|\alpha\rangle$ and $|\beta\rangle$, we can consider the following three scattering amplitudes: $\mathcal{M}(\alpha \rightarrow \alpha)$, $\mathcal{M}(\beta \rightarrow \beta)$ and $\mathcal{M}(\beta \rightarrow \alpha) = \overline{\mathcal{M}(\alpha \rightarrow \beta)}$. Here, we estimate their asymptotic behavior and translate the unitarity bound for them into constraints on the scattering amplitude $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ between the momentum eigenstates. In the estimation made below, $\mathbf{p}_1, \mathbf{p}_2$ ($\mathbf{k}_1, \mathbf{k}_2$) denote the momenta of the initial (final) two particles.

² The contributions of factors in the integral are given by $d^d p_1 \sim d^d p_2 \propto P^d$, $E_1 \sim E_2 \propto P^z$, $\delta(E_1 + E_2 - E) \propto P^{-z}$ and $\delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \propto P^{-d}$.

i. $\mathcal{M}(\alpha \rightarrow \alpha)$

In this scattering process, all the momenta of the initial and the final particles approach infinity, i.e. $|\mathbf{p}_1|, |\mathbf{p}_2|, |\mathbf{k}_1|, |\mathbf{k}_2| \approx P$. In the integral form of the scattering amplitude Eq. (34), both the initial and the final phase space factors $d\Pi(\mathbf{p})$ and $d\Pi(\mathbf{k})$ have the same high-energy behavior as N_α . Assuming that the leading order behavior of $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ on the support of h_α is $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a$, we can estimate the amplitude as

$$\mathcal{M}(\alpha \rightarrow \alpha) \approx P^{a-3z+d}. \quad (45)$$

Therefore, the unitarity bound $\mathcal{M}(\alpha \rightarrow \alpha) \leq 1$ is satisfied if

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a \quad \text{with} \quad a \leq 3z - d. \quad (46)$$

ii. $\mathcal{M}(\beta \rightarrow \beta)$

In this case, the initial and the final momenta of the first particle are small $|\mathbf{p}_1|, |\mathbf{k}_1| \propto P^0$ and those of the second particle become large $|\mathbf{p}_2|, |\mathbf{k}_2| \propto P$ for large P . Both $d\Pi(\mathbf{p})$ and $d\Pi(\mathbf{k})$ have the same high-energy behavior as N_β . Therefore, assuming that $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a$, we find that

$$\mathcal{M}(\beta \rightarrow \beta) \approx P^{a-2z+1}. \quad (47)$$

The unitarity bound $\mathcal{M}(\beta \rightarrow \beta) \leq 1$ is satisfied if

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a \quad \text{with} \quad a \leq 2z - 1. \quad (48)$$

iii. $\mathcal{M}(\beta \rightarrow \alpha)$

For the scattering from $|\beta\rangle$ to $|\alpha\rangle$, only the initial momentum of the first particle is small $|\mathbf{p}_1| \propto P^0$ and the others are large $|\mathbf{p}_2|, |\mathbf{k}_1|, |\mathbf{k}_2| \propto P$. The measures $d\Pi(\mathbf{p})$ and $d\Pi(\mathbf{k})$ have the same high-energy behaviors as N_β and N_α , respectively. Assuming that $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a$, we find that

$$\mathcal{M}(\beta \rightarrow \alpha) \approx P^{a-\frac{5z-d-1}{2}}. \quad (49)$$

The unitarity bound $\mathcal{M}(\beta \rightarrow \alpha) \leq 1$ is satisfied if

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^a \quad \text{with} \quad a \leq (5z - d - 1)/2. \quad (50)$$

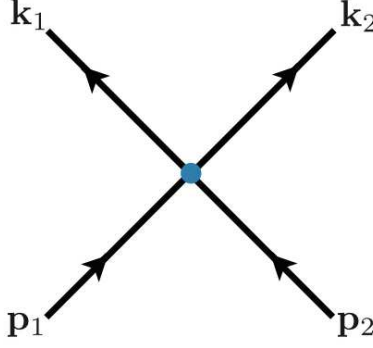


Fig. 5: four point tree-level diagram

5 Constraints for Vertices from Unitarity Bound

In this section, we see that the unitarity conditions derived in Sec.4 agree with the extended (and conventional) PCR conditions for quartic and cubic interactions.

5.1 Quartic Interactions

We first consider the following generic quartic interaction term:

$$S_4 = \int dt d^d x (\partial_x^{a_1} \phi) (\partial_x^{a_2} \phi) (\partial_x^{a_3} \phi) (\partial_x^{a_4} \phi). \quad (51)$$

Without loss of generality, we can set $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. The momentum dependence of scattering amplitude (Fig. 5) is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \propto |\mathbf{p}_1|^{a_1} |\mathbf{p}_2|^{a_2} |\mathbf{k}_1|^{a_3} |\mathbf{k}_2|^{a_4} + (\text{permutations w.r.t. } a_i), \quad (52)$$

where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1$ and \mathbf{k}_2 are the momentum of external lines. The extended PCR conditions (16) and (17) require that

$$a_1 + a_2 + a_3 + a_4 \leq 3z - d, \quad (53)$$

$$a_2 + a_3 + a_4 < (5z - d)/2, \quad \Rightarrow \quad a_2 + a_3 + a_4 \leq (5z - d - 1)/2 \quad (54)$$

$$a_3 + a_4 < 2z, \quad \Rightarrow \quad a_3 + a_4 \leq 2z - 1, \quad (55)$$

where we have rewritten the inequalities with $<$ into those with \leq by using the fact that a_1, a_2, a_3, a_4, d , and z are all integers. In the following, we show that these conditions agree with the tree-level unitarity conditions.

i. $|\alpha\rangle \rightarrow |\alpha\rangle$

For the scattering from $|\alpha\rangle$ to $|\alpha\rangle$, the momenta of all the external lines are proportional to P for large P , so that the high energy behavior of $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_1+a_2+a_3+a_4}. \quad (56)$$

Thus, the condition (46) gives the constraint

$$a_1 + a_2 + a_3 + a_4 \leq 3z - d. \quad (57)$$

This coincides with the conventional PCR condition (53).

ii. $|\beta\rangle \rightarrow |\beta\rangle$

For the scattering from $|\beta\rangle$ to $|\beta\rangle$, two external momenta are proportional to P , so that the leading term in $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_3+a_4}. \quad (58)$$

Thus, the condition (48) gives

$$a_3 + a_4 \leq 2z - 1. \quad (59)$$

This agrees with the extended PCR condition (55).

iii. $|\beta\rangle \rightarrow |\alpha\rangle$

For the scattering from $|\beta\rangle$ to $|\alpha\rangle$, three external momenta approach infinity, and thus

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_2+a_3+a_4}. \quad (60)$$

The constraint (50) gives

$$a_2 + a_3 + a_4 \leq (5z - d - 1)/2. \quad (61)$$

This is equivalent to the extended PCR condition (54).

5.2 Cubic Interactions

Next, we derive the constraints for cubic interactions from the unitarity conditions. The generic form of cubic interactions is given by

$$S_3 = \int dt d^d x (\partial_x^{a_1} \phi) (\partial_x^{a_2} \phi) (\partial_x^{a_3} \phi), \quad (62)$$

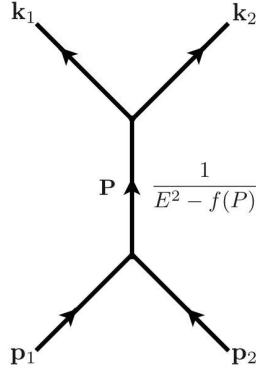


Fig. 6: s-channel diagram

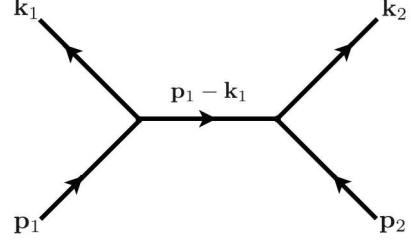


Fig. 7: t-channel diagram

with $0 \leq a_1 \leq a_2 \leq a_3$. The extended PCR condition implies that

$$a_1 + a_2 + a_3 \leq (5z - d)/2, \quad (63)$$

$$a_2 + a_3 < 2z \quad \Rightarrow \quad a_2 + a_3 \leq 2z - 1, \quad (64)$$

where we have rewritten the second inequality by using the fact that a_1, a_2, a_3, d and z are integers. For the interaction term (62), the contribution of the corresponding 3-point vertex takes the form

$$V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_2) = |\mathbf{p}_1|^{a_1} |\mathbf{p}_2|^{a_2} |\mathbf{p}_1 + \mathbf{p}_2|^{a_3} + (\text{permutations w.r.t. } a_i). \quad (65)$$

i. s-channel

Let us first consider the s-channel scattering amplitudes (Fig. 6):

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \propto \frac{V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_2) V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_1 + \mathbf{k}_2)}{E^2 - f(P^2)}. \quad (66)$$

The denominator $E^2 - f(P^2)$ can be roughly estimated as P^{2z} in the UV limit.³

When all the external momenta approach infinity ($|\alpha\rangle \rightarrow |\alpha\rangle$ scattering), the scattering amplitude can be estimated as

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{2a_1 + 2a_2 + 2a_3 - 2z}. \quad (67)$$

³ This estimation seems invalid for $E = P^z + \mathcal{O}(P^{z-1})$. Such a situation always occurs if the initial and/or final states are $|\beta\rangle$. However, as we will see in the next section, the propagator can have contributions from the imaginary part of the self-energy diagrams $\text{Im}\Sigma(E, P)$. From a simple dimensional analysis, we can find that marginal interactions can give the leading contribution to the imaginary part of the self-energy $\text{Im}\Sigma(E, P) \approx P^{2z}$.

From the condition (46), we find that

$$a_1 + a_2 + a_3 \leq (5z - d)/2. \quad (68)$$

For the scattering from $|\beta\rangle$ to $|\beta\rangle$, the leading part of the scattering amplitude is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{2a_2+2a_3-2z}. \quad (69)$$

The condition (48) implies that

$$a_2 + a_3 \leq 2z - \frac{1}{2} \quad \Rightarrow \quad a_2 + a_3 \leq 2z - 1, \quad (70)$$

where we have used the fact that all a_i and z are integers. These two conditions are equivalent to the extended PCR conditions (63) and (64).

For the scattering from $|\beta\rangle$ to $|\alpha\rangle$, the leading order behavior of the amplitude becomes

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_1+2a_2+2a_3-2z}. \quad (71)$$

The condition (50) implies that

$$a_1 + 2a_2 + 2a_3 \leq (9z - d - 1)/2. \quad (72)$$

This constraint is automatically satisfied if both Eqs. (68) and (70) are met.

ii. t-(and u-)channels

Next, we consider the t-(and u-)channel scattering amplitude (Fig. 7). In most cases, the magnitude of internal momentum $|\mathbf{p}_1 + \mathbf{p}_2|$ becomes of order $\mathcal{O}(P)$ and the estimation is the same as that in the case of the s-channel. With a fine tuning of the initial and final states, the internal momentum can be of order $\mathcal{O}(P^0)$ in the limit $P \rightarrow \infty$. We show that such scattering amplitudes have different asymptotic behaviors but give no additional constraint for the cubic interactions.

For the scattering from $|\alpha\rangle$ to $|\alpha\rangle$, the leading behavior of the scattering amplitude can be estimated as

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{2a_2+2a_3}. \quad (73)$$

The condition (46) implies that

$$a_2 + a_3 \leq (3z - d)/2. \quad (74)$$

Since the right hand side is smaller than or equal to $(2z - 1)$, this condition is weaker than the inequality (70). For the scattering from $|\beta\rangle$ to $|\beta\rangle$, the leading behavior of the scattering amplitude is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_2+a_3}. \quad (75)$$

From the condition (48), we obtain the constraint

$$a_2 + a_3 \leq 2z - 1. \quad (76)$$

This is the same condition as the inequality (70). If momenta of three external lines approach infinity, the momentum of internal line cannot be finite. In summary, these types of scattering amplitudes gives no additional constraint for the cubic interactions.

6 On-shell Intermediate States and Resonances

In Lifshitz-type theories, there exists a peculiar phenomenon in scattering processes which is absent in relativistic theories, namely the appearance of on-shell intermediate states. When the intermediate particle has an on-shell momentum, the high-energy s-channel amplitude appears to violate the unitarity bound from the naive estimate in the previous section. However, in such cases, the tree-level discussion breaks down and we should correctly take into account the corresponding resonance effects. In this section, we show that the unitarity is not violated if all the extended PCR conditions are satisfied.

6.1 Resonances Induced by Identical Particles

In relativistic theories, any single particle cannot decay into two (or more) identical particles. In contrast, in Lifshitz-type theories, single particle states can decay into multi-particle states if its momentum is larger than a certain stability threshold. This can be seen from the dispersion relation given in Eq.(3). For two-particle states, the total energy and momentum satisfy the following inequality:

$$E_{two} \geq E_{min} := 2\sqrt{f(P^2/4)}, \quad (77)$$

where the equality holds only when two particles have the same momentum, i.e.

$$\mathbf{p}_1 = \mathbf{p}_2 = \frac{1}{2}\mathbf{P}. \quad (78)$$

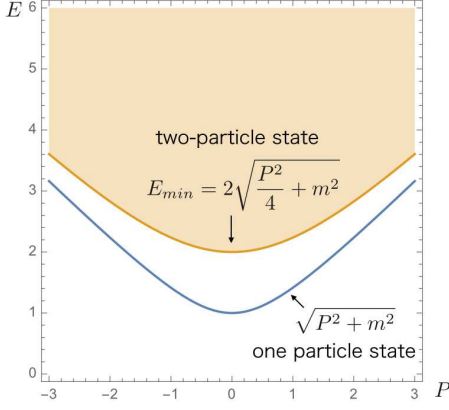


Fig. 8: Spectrum for $z=1$ and $f(p^2) = p^2 + m^2$

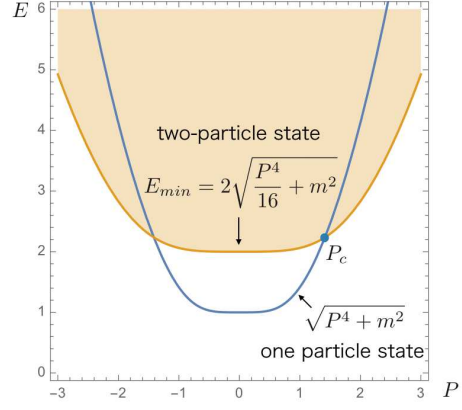


Fig. 9: Spectrum for $z=2$ and $f(p^2) = p^4 + m^2$

A single particle can decay into two particles, if its energy E_{one} is larger than E_{min} . In the Lifshitz scaling theory with $z > 1$, the difference $E_{min} - E_{one}$ for $P = 0$ has the positive value $\sqrt{f(0)}$, whereas it becomes negative for large P (see Fig. 8 and Fig. 9):

$$E_{min} - E_{one} = \left(\frac{1}{2^{z-1}} - 1 \right) P^z + \mathcal{O}(P^{z-1}) < 0. \quad (79)$$

Therefore, there exists a stability threshold P_c at which $E_{min} - E_{one} = 0$. This also implies that on-shell intermediate states can appear in the s-channel scattering processes if the total momentum is greater than P_c .

When there exists an on-shell intermediate state, we need to take into account the effect of the resonance⁴. In terms of the tree-level propagator G_0 and the 1PI self-energy $-i\Sigma(E, P)$, the full propagator G is given by

$$G = G_0 + G_0(-i\Sigma)G_0 + G_0(-i\Sigma)G_0(-i\Sigma)G_0 + \dots = \frac{i}{E^2 - f(P^2) - \Sigma}, \quad (80)$$

where we have used the tree-level propagator derived from the quadratic action (2);

$$G_0 = \frac{i}{E^2 - f(P^2)}. \quad (81)$$

Imposing the on-shell renormalization condition

$$\text{Re } \Sigma(E, P) \Big|_{E=\sqrt{f(P^2)}} = 0, \quad (82)$$

we find that the propagator at $E = \sqrt{f(P^2)}$ is given by the imaginary part of the self-energy Σ :

$$G(E = \sqrt{f(P^2)}, P) = - \frac{1}{\text{Im } \Sigma(E, P)} \Big|_{E=\sqrt{f(P^2)}}. \quad (83)$$

Therefore, the propagator does not diverge at $E = \sqrt{f(P^2)}$.

⁴ In theories with the Lorentz symmetry, resonance occurs only at a finite Mandelstam variable s , so that we do not need to consider the resonance in the discussion of the unitarity bound for high energy scattering amplitudes.

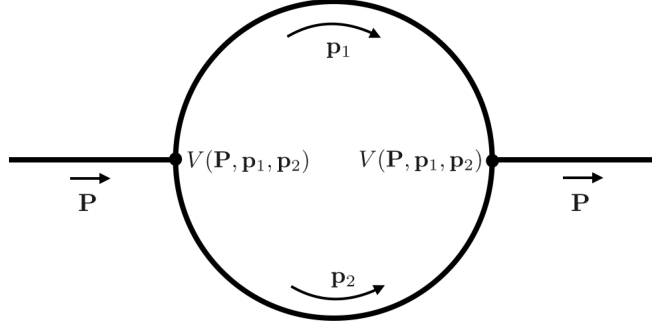


Fig. 10: Self energy diagram

6.2 s-channel Amplitudes with On-shell Internal States

Let us estimate the s-channel amplitudes with the on-shell internal propagator and see that the unitarity condition is satisfied if all conditions shown in Sec. 4 are obeyed.

The momentum dependence of the self-energy can be estimated as follows. The leading order contribution to the 1PI self energy $-i\Sigma(E, P)$ is give by the one-loop diagram (see Fig. 10), which takes the form

$$-i\Sigma_{1\text{-loop}}(E, P) = i \int d\omega d^d p |V(\mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)|^2 G_1 G_2, \quad (84)$$

with $\mathbf{p}_1 := \mathbf{P}/2 + \mathbf{p}$, $\mathbf{p}_2 := \mathbf{P}/2 - \mathbf{p}$ and

$$G_1 = \frac{1}{(E/2 + \omega)^2 - f(p_1^2) + i\epsilon}, \quad G_2 = \frac{1}{(E/2 + \omega)^2 - f(p_2^2) + i\epsilon}. \quad (85)$$

Here, $V(\mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ is a three-point vertex, for which we use the generic form (62) for definiteness. To evaluate the imaginary part of Σ , we use the following relation:

$$\frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x), \quad (86)$$

where $\mathcal{P}\frac{1}{x}$ denotes the principal value of $1/x$. Then, the imaginary part of Σ can be rewritten as

$$\begin{aligned} \text{Im } \Sigma(E, P) & \sim \pi^2 \int d\omega d^d p |V(\mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)|^2 \delta((E/2 + \omega)^2 - f(p_1^2)) \delta((E/2 + \omega)^2 - f(p_2^2)) \\ & = \pi^2 \int \frac{d^d p_1}{2E_1} \frac{d^d p_2}{2E_2} |V(\mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)|^2 \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}), \end{aligned} \quad (87)$$

where we have used the fact that the product of the principal values do not contribute the imaginary part. Note that this is the cutting rule which relates the imaginary part of the

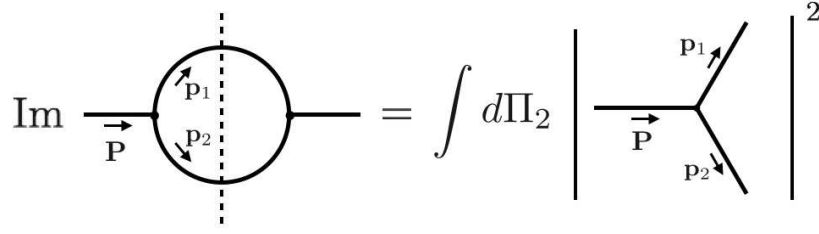


Fig. 11: Cutting rule

one-loop self energy diagram and the tree-level decay diagram (see Fig.11). Suppose that the interaction term (62) gives the leading order contribution among cubic order interactions. Then, $\text{Im } \Sigma(E, P)$ can be written as

$$\text{Im } \Sigma(E, P) \sim \pi^2 \int d\Pi_2(\mathbf{p}) \left| p_1^{a_1} p_2^{a_2} |\mathbf{p}_1 + \mathbf{p}_2|^{a_3} + (\text{permutations w.r.t. } a_i) \right|^2. \quad (88)$$

There are two regions in the integration domain either of which can give a leading contribution to $\text{Im } \Sigma$:

1. The region where both p_1 and p_2 are of order $\mathcal{O}(P)$,
2. The region where only p_1 (or p_2) is of order $\mathcal{O}(P)$.

The contributions from region 1 and 2 are respectively estimated as P^{γ_1} and P^{γ_2} with

$$\gamma_1 := d - 3z + 2(a_1 + a_2 + a_3), \quad \gamma_2 := -2z + 1 + 2(a_2 + a_3), \quad (89)$$

where the estimation of the delta function for the latter case is the same as that in the discussion below Eq. (43). Therefore, the asymptotic behavior of $\text{Im } \Sigma$ is given by

$$\text{Im } \Sigma \approx P^{\max(\gamma_1, \gamma_2)}. \quad (90)$$

Let us see the s-channel diagram Fig. 6 with an on-shell intermediate state. Since the interaction term Eq. (62) gives the leading contribution by assumption, diagrams constructed with the other 3-point vertices are sub-leading. Therefore, it is sufficient to check the unitarity for the s-channel amplitudes constructed with the vertices corresponding to Eq. (62).

i. $|\alpha\rangle \rightarrow |\alpha\rangle$

For $\gamma_1 \geq \gamma_2$, the asymptotic behavior of $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{3z-d}. \quad (91)$$

This satisfies the unitarity condition (46). For $\gamma_1 \leq \gamma_2$, the amplitude $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ can be estimated as

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{2a_1-1+2z}. \quad (92)$$

Since the assumption $\gamma_1 \leq \gamma_2$ implies that $2a_1 - 1 + 2z \leq 3z - d$, the unitarity condition (46) is satisfied.

ii. $|\beta\rangle \rightarrow |\beta\rangle$

For $\gamma_1 \geq \gamma_2$, the leading term in $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ is given by

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{-d+3z-2a_1}. \quad (93)$$

Since $-d + 3z - 2a_1 \leq 2z - 1$ for $\gamma_1 \geq \gamma_2$, the unitarity condition (48) is satisfied. For $\gamma_1 \leq \gamma_2$, we can estimate $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ as

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{2z-1}. \quad (94)$$

This amplitude satisfies the unitarity condition (48).

iii. $|\alpha\rangle \rightarrow |\beta\rangle$

For $\gamma_1 \geq \gamma_2$, the leading behavior of the amplitude can be estimated as

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{-d+3z-a_1}. \quad (95)$$

Since the inequality $\gamma_1 \geq \gamma_2$ gives $-d + 3z - a_1 \leq \frac{5z-d-1}{2}$, this satisfies the unitarity condition (50). For $\gamma_1 \leq \gamma_2$, the leading term of $M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2)$ reads

$$M(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{k}_1, \mathbf{k}_2) \approx P^{a_1+2z-1}. \quad (96)$$

Since the inequality $\gamma_1 \leq \gamma_2$ can be rewritten as $a_1 - 1 + 2z \leq \frac{5z-d-1}{2}$, the condition (50) is satisfied.

7 Summary

In this paper, we have studied the renormalizability and unitarity of the self-interacting Lifshitz scalar theory. Because of the anisotropic scaling between the space and time directions, the conditions for the renormalizability and unitarity are different from those in the relativistic

theories. We have derived the conditions for the cubic and quartic interaction terms and shown that the renormalizability and unitarity require equivalent conditions. This is one of the evidences for the equivalence between the renormalizability and unitarity in generic field theories.

We have shown in Sec.2 that the condition for the renormalizability is extended from the conventional one which we usually apply to the relativistic theory. Because of the anisotropic scaling, the dimension of the Lifshitz scalar field can be negative. In such cases, even if an interaction term satisfies the conventional PCR condition (16), a part of the term can have a large dimension which exceeds the limit for a renormalizable interaction. If only the part of the operator contributes to a loop integral, it behaves like a non-renormalizable term in the sense of the conventional PCR, i.e. it increases the degree of divergence of the loop integral. Therefore, such a type of interaction term is non-renormalizable. We have derived the constraints that rule out this type of interaction term. We call these conditions the extended PCR conditions. Their explicit forms are given as inequalities for the dimensions of coupling constants (14) and (15) (or for the dimensions of interaction terms (16) and (17)).

We have also derived the conditions for cubic and quartic interactions from the tree-level unitarity bound for two-particle scattering amplitudes. Unlike in the relativistic case, scattering amplitudes have the reference frame dependence, a variety of conditions are required for unitarity. We have seen in Sec.5 that the conditions for unitarity are equivalent to the extended PCR conditions. We have also investigated, in Sec.6, the cases where the intermediate propagator in the s-channel diagram satisfies the on-shell condition. The unitarity bound for the s-channel diagram with the on-shell intermediate propagator appears to give stronger conditions, but actually the tree-level argument is not applicable in such situations. We have shown in Sec.6 that if the nonlinear effect, that is resonance, is taken into account, interaction terms satisfying the extended PCR conditions do not violate the unitarity bound.

From our results, the equivalence between renormalizability and unitarity is expected to hold true even in non-relativistic theories. This relation is helpful to investigate the renormalizability of the HL gravity. Moreover, the extended PCR condition gives constraints on the form of the action for matter fields in the HL gravity. To preserve the anisotropic scaling property of the HL gravity, matter fields are also expected to follow the same anisotropic scaling. The dimensions of fields are zero for $d = z = 3$, and the fields have to satisfy the extended PCR. Meanwhile, for the gravitational action in the HL gravity with $d = z = 3$, the conventional PCR can still work because of the symmetry. The invariance under the foliation preserving diffeomorphism constrains the form of the possible gravitational operators, and all of their dimensions are positive in the HL gravity with $d = z = 3$. Without non-positive dimensional operator, the conventional PCR is

enough for the renormalization. It may be possible to see the cancellation among the bad-behaved diagrams both for renormalizability and for unitarity in the same way as the Weinberg-Salam theory. In the HL gravity with $d = 3$ and $z \geq 7$, the three dimensional curvature has non-positive dimension, and thus the extended PCR would be required for renormalization. To investigate the relevance of symmetries to renormalizability and unitarity, it is important to extend our study to Lifshitz-type theories with symmetries such as non-linear sigma models [15, 16, 17, 18].

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A Renormalizability of Counter Terms

A.1 Extended PCR of Counter Terms

Here, we show that, if the extended PCR conditions (16) and (17) hold, the required counter terms also satisfy Eqs.(16) and (17). As in the discussion in Sec.2, the counter term for an n -point diagram with L loops, P propagators and V vertices can be estimated as

$$\prod_{i=1}^V \prod_{\bar{m}} \left(\partial^{a_{\bar{m}}^i} \phi \right) \int (d\omega d^d p)^L \left(\frac{1}{\omega^2 - p^{2z}} \right)^P \prod_i \left(\prod_m p^{a_m^i} \right), \quad (97)$$

where i denotes the label of vertices and m and \bar{m} are the labels for $\partial^{a_m^i} \phi$ in Eq. (6) corresponding to the internal and external lines at each vertex. The degree of divergence of the integral can be estimated as

$$D = z + d - \sum_i \left(z + d - \sum_m (a_m^i + [\phi]) \right). \quad (98)$$

Here we have used $[\lambda] = z + d - \sum_{l=1}^n (a_l + [\phi])$. Since we are interested in the divergent contributions, we consider the case where D is non-negative.

The integral in Eq. (97) has sub-leading divergent contributions. We expand it with respect to the small value k/p where k is the magnitude of external momenta. Expanding the integral, we find that the marginal terms in the counter term have totally D extra spatial derivatives. Therefore, the extended PCR conditions for the counter terms are written as

$$\sum_i \sum_{\text{all } \bar{m}} (a_{\bar{m}}^i + [\phi]) + D \leq z + d, \quad (99)$$

$$\sum_i \sum_{\text{part of } \bar{m}} (a_{\bar{m}}^i + [\phi]) + D < z + d. \quad (100)$$

Substituting Eq. (98) into the left hand side of the first inequality, we have

$$\sum_i \sum_{\text{all } \bar{m}} (a_{\bar{m}}^i + [\phi]) + D = z + d + \sum_i \left[-(z + d) + \sum_{l=1}^n (a_l^i + [\phi]) \right]. \quad (101)$$

This implies that the inequality (99) is met, when all the vertices satisfy the conventional PCR condition $\sum_{l=1}^n (a_l^i + [\phi]) \leq z + d$. Similarly, we can show that the inequality (100) holds if the extended conditions PCR are satisfied for all vertices.

A.2 Number of Counter Terms

If the dimension of a field is non-positive, we have an infinite number of terms satisfying the extended PCR. For instance, the interaction term $S_{int} = \int dt d^d x \phi^n$ satisfies the extended PCR for any n . It seems that an infinite number of counter terms are required. However, we show here that, if the number of terms in the bare action is finite, we can renormalize the theory with a finite number of counter terms.

We consider the following generic interaction term

$$S_{m,\mathcal{D}} = \int dt d^d x (\partial_x^{a_1} \phi) \dots (\partial_x^{a_m} \phi) (\partial_x^{b_1} \phi) \dots (\partial_x^{b_n} \phi). \quad (102)$$

Here, we assume that the first m $(\partial_x^{a_j} \phi)$'s have non-positive dimensions while the dimensions of the last n $(\partial_x^{b_j} \phi)$'s are positive, that is,

$$[\partial_x^{a_j} \phi] < 0 \quad (j = 1, \dots, m), \quad [\partial_x^{b_j} \phi] > 0 \quad (j = 1, \dots, n). \quad (103)$$

The number \mathcal{D} denotes the dimension of the product $(\partial_x^{b_1} \phi) \dots (\partial_x^{b_n} \phi)$, that is, for an interaction term $S_{int,m,\mathcal{D}}$, the index m indicates the number of $(\partial_x^{a_j} \phi)$'s with non-positive dimensions and \mathcal{D} shows the total dimension of $(\partial_x^{b_j} \phi)$'s with positive dimensions. From the extended PCR conditions, \mathcal{D} has to satisfy

$$\mathcal{D} \leq d + z, \quad (\text{for } m = 0). \quad (104)$$

$$\mathcal{D} < d + z, \quad (\text{for } m \geq 1). \quad (105)$$

The second inequality (105) can also be rewritten in

$$\mathcal{D} \leq d + z - 1/2, \text{ (for } m \geq 1\text{)}. \quad (106)$$

Since b_j , d and z are all integers, $[\partial_x^{b_j} \phi]$ has to be a half of integer, that is, $[\partial_x^{b_j} \phi] \geq \frac{1}{2}$. Because $\mathcal{D}(\geq n/2)$ is bounded both from above, n should also be bounded from above; $n \leq 2(d + z) - 1$. Therefore the form of the part $(\partial_x^{b_1} \phi) \cdots (\partial_x^{b_n} \phi)$ is limited.

We consider a generic loop diagram constructed with p interaction terms $S_{m_1, \mathcal{D}_1}, \dots, S_{m_p, \mathcal{D}_p}$ with nonzero m_j and q interaction terms $S_{m_{p+1}, \mathcal{D}_{p+1}}, \dots, S_{m_{p+q}, \mathcal{D}_{p+q}}$ with $m_{p+j} = 0$. The maximum divergence occurs when all positive dimensional operators appear as the internal lines while negative dimensional operators correspond to external lines. Dimension-zero operators can be either internal lines or external lines. Then, the leading order behavior of the UV divergence can be estimated as

$$D = d + z - \sum_{j=1}^p (d + z - \mathcal{D}_j) - \sum_{j=1}^q (d + z - \mathcal{D}_{p+j}). \quad (107)$$

In general, other counter terms are required for sub-leading contributions, which appear when positive dimensional operators become external lines, or when loop diagrams are expanded with respect to external momenta. However, the loop integrals become finite if enough many positive dimensional operators become external lines, or if there are enough higher order contributions from the expansion.

We consider a divergent loop diagram in which some positive dimensional operators appear in the external lines. Suppose that S_{m_c, \mathcal{D}_c} is the corresponding counter term and \mathcal{D}_c is the sum of dimensions of the positive dimensional operators. The degree of divergence is read as

$$D = d + z - \mathcal{D}_c - \sum_{j=1}^p (d + z - \mathcal{D}_j) - \sum_{j=1}^q (d + z - \mathcal{D}_{p+j}). \quad (108)$$

Since $D \geq 0$ for divergent diagrams, counter terms cannot have arbitrarily large \mathcal{D}_c . The bound for \mathcal{D}_c can be found from inequalities (104) and (106) as

$$0 \leq D \leq \begin{cases} \mathcal{D}_k - \mathcal{D}_c - \frac{p-1}{2} & (\text{for } k \leq p \text{ and } p \geq 1) \\ \mathcal{D}_k - \mathcal{D}_c - \frac{p}{2} & (\text{for } k \geq p + 1) \end{cases}, \quad (109)$$

where we have replaced \mathcal{D}_j with their maximum values except for $\mathcal{D}_{j=k}$. Thus, \mathcal{D}_c has to be smaller than or equal to \mathcal{D}_k , and hence all counter terms satisfy the extended PCR condition, as we have seen in Appendix A.1.

The value of p is also suppressed by inequality (109). Thus, m_c is bounded from above as

$$m_c \leq \sum_{i=1}^{p_{max}} m_i \leq m_{max} p_{max}, \quad (110)$$

where m_{max} and p_{max} are the maximum value of m_i and p , respectively. When a counter term S_{m_c, \mathcal{D}_c} is not included in the original bare action, it has to be added to the modified action. Then, if the counter term has $m_c > m_{max}$, the maximum value m_{max} is also modified as $m_{max}^{modified} = m_c$.

In the modified theory, the added counter terms may give rise to new divergences and we may need to iteratively add new counter terms with larger m_c . We can show that this operation terminates after finite iterations as follows. Counter terms with $m_c > m_{max}$ are required to cancel divergences from diagrams with $p \geq 2$. We can see from inequality (109) that $\mathcal{D}_c \geq \mathcal{D}_k - \frac{1}{2}$. This implies that \mathcal{D}_c decreases by the iterative operation, but \mathcal{D}_c has to be positive. Thus, the operation stops and the counter terms in the final operation have only non-positive dimensional operators. As a result, m_c is bounded from above.

We check that loop contributions constructed with the operator having only non-positive dimensional operators require a finite number of counter terms. We consider operators with $q = 0$, i.e. all operators have non-positive dimensions. By assumption, the dimension of each coupling constant must be larger than $d + z$, that is,

$$[\lambda_j] > d + z. \quad (111)$$

The degree of divergence of a loop diagram can be estimated as

$$D = [\lambda_c] - \sum_{j=1}^n [\lambda_j], \quad (112)$$

where $[\lambda_c]$ is the dimension of the coupling constant in the required counter terms. If the counter terms are required, D is equal to or larger than zero, and then from (111), $[\lambda_c]$ is suppressed as

$$[\lambda_c] \leq [\lambda_j]. \quad (113)$$

Therefore, the number of counter terms for loop diagrams constructed with $q = 0$ is finite.

Since \mathcal{D}_c and m_j are bounded from above and a finite number of counter terms are required for the loop diagram constructed with $q = 0$, the number of counter terms are finite if the original action has a finite number of terms.

B Evaluation of Phase Space Integrals

In this section, we derive the asymptotic forms of the phase space integrals used in this paper. We assume that the dispersion relation takes the form

$$E = f(p^2) = p^{2z} + \eta^{2z} \sum_{i=0}^{n-1} c_{2i} \left(\frac{p}{\eta} \right)^{2i}, \quad (114)$$

where η is a scale parameter with $[\eta] = 1$ and c_{2i} are dimensionless parameters. Let us consider the integral of the form

$$X = \int_I \frac{d^d p_1}{2E_1} \frac{d^d p_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{P}, \eta), \quad (115)$$

where $F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{P}, \eta)$ is a function of the momenta and the scale parameter which does not depend on other dimensionful parameters. We focus on the subspace of two particle Hilbert space where the total energy and momentum are related as

$$E = P^z + \mathcal{O}(P^{z-1}). \quad (116)$$

Let us first consider the integral corresponding to the state $|\alpha\rangle$, for which the domain of integration is given by

$$I = I_\alpha = \{ (\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid ||\mathbf{p}_1| - |\mathbf{p}_2|| \leq P/2 \}. \quad (117)$$

Changing the integration variable as $\mathbf{p}_i = P\mathbf{u}_i$ and assuming that $[F] = a$, i.e.

$$F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{P}, \eta) = P^a F(\mathbf{u}_1, \mathbf{u}_2, \hat{\mathbf{P}}, \eta/P), \quad (\hat{\mathbf{P}} = \mathbf{P}/P), \quad (118)$$

we find that for small $\delta \equiv \eta/P$, the asymptotic form of X is given by

$$X = P^{d-3z+a} \left[\int_{I'_\alpha} \frac{d^d u_1 d^d u_2}{4u_1^z u_2^z} \delta(u_1^z + u_2^z - 1) \delta^d(\mathbf{u}_1 + \mathbf{u}_2 - \hat{\mathbf{P}}) F(\mathbf{u}_1, \mathbf{u}_2, \hat{\mathbf{P}}, 0) + \mathcal{O}(\delta) \right], \quad (119)$$

where the domain of integration is $I'_\alpha = \{ (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid ||\mathbf{u}_1| - |\mathbf{u}_2|| \leq 1/2 \}$. Since the leading term in Eq. (119) is independent of P , the asymptotic form of X can be estimated as

$$X \approx \text{const.} \times P^{d-3z+a}. \quad (120)$$

Next, let us consider the integral corresponding to the state $|\beta\rangle$, for which the domain of integration is given by

$$I = I_\beta = \{ (\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid |\mathbf{p}_1| \leq \epsilon \equiv c\eta \}, \quad (121)$$

where c is a dimensionless parameter. Integrating over \mathbf{p}_2 and changing the integration variable as $\mathbf{p}_1 = \eta\mathbf{v}$, we find that the asymptotic form of the energy conservation factor is given by

$$E_1 + E_2 - E \approx \eta P^{z-1} \left[-z \hat{\mathbf{P}} \cdot \mathbf{v} + \mathcal{O}(\delta) \right], \quad (\text{for } z > 1). \quad (122)$$

Assuming that the asymptotic form of the function F takes the form

$$\eta^{-a} F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{P}, \eta) \approx \frac{P^b}{\eta^b} \left(v^q \tilde{F}(\theta) + \mathcal{O}(\delta) \right), \quad ([F] = a, \quad \hat{\mathbf{P}} \cdot \mathbf{v} = v \cos \theta), \quad (123)$$

we can approximate the integral as

$$X = \eta^a \int_{|\mathbf{v}| < c} d^d v \frac{P^b / \eta^b \left(v^q \tilde{F}(\theta) + \mathcal{O}(\delta) \right)}{\sqrt{v^{2z} + \sum c_{2i} v^{2i}} \sqrt{P^{2z} + \mathcal{O}(\delta)}} \delta \left(\eta P^{z-1} \left[-z \hat{\mathbf{P}} \cdot \mathbf{v} + \mathcal{O}(\delta) \right] \right). \quad (124)$$

Therefore, the leading term in X is given by

$$X \approx \text{const.} \times P^{1-2z+b}, \quad \left(\text{const.} = \eta^{a-b-1} \int_{|\mathbf{v}| < c} \frac{v^q \tilde{F}(\theta) d^d v}{\sqrt{v^{2z} + \sum c_{2i} v^{2i}}} \right). \quad (125)$$

References

- [1] C. H. Llewellyn Smith, Phys. Lett. B **46**, 233 (1973).
- [2] J. M. Cornwall, D. N. Levin and G. Tiktopoulos, Phys. Rev. Lett. **30**, 1268 (1973) [Phys. Rev. Lett. **31**, 572 (1973)].
- [3] J. M. Cornwall, D. N. Levin and G. Tiktopoulos, Phys. Rev. D **10**, 1145 (1974) [Phys. Rev. D **11**, 972 (1975)].
- [4] F. A. Berends and R. Gastmans, Nucl. Phys. B **88**, 99 (1975).
- [5] E. Lifshitz, On the theory of Second-Order Phase transitions I II Zh. Eksp. Teor. Fiz. 11 (1941) 255, 269.
- [6] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
P. Horava, JHEP **0903**, 020 (2009) [arXiv:0812.4287 [hep-th]].
- [7] K. Izumi, T. Kobayashi and S. Mukohyama, JCAP **1010**, 031 (2010) [arXiv:1008.1406 [hep-th]].
- [8] T. Fujimori, T. Inami, K. Izumi and T. Kitamura, Phys. Rev. D **91**, no. 12, 125007 (2015) [arXiv:1502.01820 [hep-th]].
- [9] D. Anselmi and M. Halat, Phys. Rev. D **76**, 125011 (2007) [arXiv:0707.2480 [hep-th]].
D. Anselmi, JHEP **0802**, 051 (2008) [arXiv:0801.1216 [hep-th]].
D. Anselmi, Annals Phys. **324**, 874 (2009) [arXiv:0808.3470 [hep-th]].
D. Anselmi, Annals Phys. **324**, 1058 (2009) [arXiv:0808.3474 [hep-th]].
- [10] M. Visser, Phys. Rev. D **80**, 025011 (2009) [arXiv:0902.0590 [hep-th]].
M. Visser, arXiv:0912.4757 [hep-th].

- [11] M. Colombo, A. E. Gumrukcuoglu and T. P. Sotiriou, arXiv:1410.6360 [hep-th].
M. Colombo, A. E. Gumrukcuoglu and T. P. Sotiriou, arXiv:1503.07544 [hep-th].
- [12] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
P. Horava, JHEP **0903**, 020 (2009) [arXiv:0812.4287 [hep-th]].
- [13] S. Mukohyama, Class. Quant. Grav. **27**, 223101 (2010) [arXiv:1007.5199 [hep-th]].
- [14] S. Mukohyama, Phys. Rev. D **80**, 064005 (2009) [arXiv:0905.3563 [hep-th]].
K. Izumi and S. Mukohyama, Phys. Rev. D **81**, 044008 (2010) [arXiv:0911.1814 [hep-th]].
K. Izumi and S. Mukohyama, Phys. Rev. D **84**, 064025 (2011) [arXiv:1105.0246 [hep-th]].
- [15] S. R. Das and G. Murthy, “CP**(N-1) Models at a Lifshitz Point,” Phys. Rev. D **80**, 065006 (2009) [arXiv:0906.3261 [hep-th]].
- [16] K. Anagnostopoulos, K. Farakos, P. Pasipoularides and A. Tsapalis, “Non-Linear Sigma Model and asymptotic freedom at the Lifshitz point,” arXiv:1007.0355 [hep-th].
- [17] P. R. S. Gomes, P. F. Bienzobaz and M. Gomes, “Competing interactions and the Lifshitz-type Nonlinear Sigma Model,” Phys. Rev. D **88**, 025050 (2013) [arXiv:1305.3792 [hep-th]].
- [18] T. Fujimori and M. Nitta, JHEP **1510**, 021 (2015) [arXiv:1507.06456 [hep-th]].